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COMMENT

Eight-vertex model and Ising model in a non-zero magnetic field: honeycomb lattice

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Abstract. The known equivalence of the honeycomb eight-vertex model with an Ising model in a non-zero magnetic field is derived via a direct mapping. Compared with a previous derivation which uses the generalised weak-graph transformation, the new method is simpler and more direct, and can be extended to other considerations.

The eight-vertex model on the honeycomb lattice is a general lattice model playing the role of the 16-vertex model for the square lattice. The honeycomb problem was first considered by Wu [1], who used a generalised weak-graph transformation [2–4] to study its soluble cases. The honeycomb eight-vertex model has since proven to be a useful tool in deducing exact results for a number of physical problems. They include the obtaining of a closed-form expression for the critical frontier of the antiferromagnetic Ising model [5], the establishment of the effect of three-body interactions on the critical behaviour of the coexistence curve diameter of a lattice gas [6], the determination of the exact phase diagram of a spin system with two- and three-site interactions [7] and an exact analysis of the spin-1 Blume–Emery–Griffiths model [8]. A key step in all these studies is the use of the aforementioned equivalence of the eight-vertex model with an Ising model in a non-zero magnetic field. While it is fairly easy to deduce this equivalence for a special subspace of the eight-vertex model, the general equivalence of the two problems is by no means obvious. In fact, it was after considerable algebraic manipulation using a generalised weak-graph transformation that the equivalence was previously established [1, 8]. In this comment we present an alternative analysis of the eight-vertex model to arrive at the same result. The new method is very simple and direct, and can be extended to other considerations.

Consider a honeycomb lattice and draw bonds along its edges such that each edge is independently 'traced' or left 'open'. Then, there are eight different vertex configurations occurring at a vertex, which we show in figure 1. With each configuration we associate a vertex weight a, b, c or d and, as in [1], we assume all weights to be positive. The partition function of the eight-vertex model is the generating function

$$Z = Z(a, b, c, d) = \sum a^{n_0} b^{n_1} c^{n_2} d^{n_3} \tag{1}$$

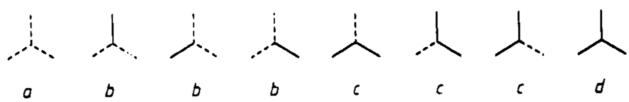

Figure 1. Vertex configurations and weights for the symmetric eight-vertex model.

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where the summation is over all bond configurations of the lattice, and n_i is the number of vertices having i bonds.

Our proof that the partition function (1) is, in fact, that of an Ising model, consists of two steps. We first formulate the eight-vertex model as a decorated Ising model, and then decimate the decorating sites. The situation is illustrated in figure 2.

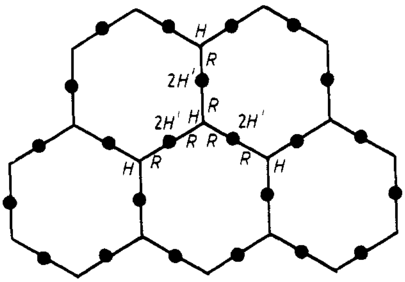

Figure 2. A decorated honeycomb lattice with the decorating sites denoted by full circles.

To formulate the eight-vertex model as a decorated Ising system, we place on each edge (of the honeycomb lattice) a decorating Ising spin σ, and let $\sigma = 1$ correspond to the edge being empty and $\sigma = -1$ correspond to the edge being occupied. Then we can describe the configuration of a vertex by specifying the configurations of the three surrounding spins. It is then possible to realise the vertex weights by introducing Ising interactions R, and magnetic fields H and $2H'$ to the decorated honeycomb lattice as shown in figure 2. The tracing of a spin at a honeycomb lattice site then leads to the following realisation:

$$\begin{aligned} a &= F\,e^{3H'}\cosh(H+3R) \qquad & b &= F\,e^{H'}\cosh(H+R) \\ c &= F\,e^{-H'}\cosh(H-R) \qquad & d &= F\,e^{-3H'}\cosh(H-3R). \end{aligned} \tag{2}$$

Here F is an overall factor which does not concern us. Solving (2) for F, R, H, H', we find

$$\begin{aligned} \cosh 2R &= B/2(AC)^{1/2} \\ e^{4H'} &= C/A \\ \cosh 2H &= \frac{2bc}{\sqrt{AC}}\left(\frac{B^2}{4AC} - \frac{B}{4bc} - 1\right) \end{aligned} \tag{3}$$

where†

$$\begin{aligned} A &\equiv bd - c^2 = F^2 e^{-2H'} \sinh^2 2R \\ B &\equiv ad - bc = 2F^2 \cosh 2R \sinh^2 2R \\ C &\equiv ac - b^2 = F^2 e^{2H'} \sinh^2 2R. \end{aligned} \tag{4}$$

Our next step is to decimate the decorating Ising spins, i.e. to replace the sequence of two R interactions with a magnetic field $2H'$ at the centre site, by a single interaction K with a magnetic field h at the two end sites. This decimation completes the mapping,

† The definition of A given here differs in sign from that used in [8].

and gives rise to a honeycomb Ising model with nearest-neighbour interactions K and a magnetic field

$$L = H + 3h \tag{5}$$

where H has been given in (3), and K and h are obtained from

$$\begin{aligned} f\,\mathrm{e}^{K+2h} &= \cosh(2H' + 2R) \\ f\,\mathrm{e}^{K-2h} &= \cosh(2H' - 2R) \\ f\,\mathrm{e}^{-K} &= \cosh 2H'. \end{aligned} \tag{6}$$

Here, f is another overall factor which does not concern us. Solving (6) for f, K and h, we obtain

$$\begin{aligned} \mathrm{e}^{4K} &= 1 + (B^2 - 4AC)/(A+C)^2 > 0 \\ \mathrm{e}^{4h} &= \cosh(2H' + 2R)/\cosh(2H' - 2R). \end{aligned} \tag{7}$$

Expressions (3), (5) and (7) now complete the description of the Ising parameters K and L.

The expression for e^{4K} in (7) is the same as that in [1]. However, as shown in [8], the sign of e^{2K} can be either positive or negative. The negation of e^{2K}, however, corresponds to the change $K \to K + \mathrm{i}\pi/2$ or $\tanh K \to 1/\tanh K$, reflecting an intrinsic symmetry of the eight-vertex model. We shall therefore disregard such sign differences in our considerations. Particularly, we consider K being real, $B > 0$, $AC > 0$. We now determine the nature of the magnetic field $L = H + 3h$.

Ferromagnetic Ising model ($K > 0$). This is the case $B^2 > 4AC$. From (3) we see that both H' and R are real so that, using (7), h is also real. Consider next $\cosh 2H$ given by (3). Since this expression essentially contains two independent variables, it is convenient to parametrise by introducing $x = a/b$, $y = d/c$, $z = b/c$ which rewrite (3) as

$$\cosh 2H = \frac{1}{2\sqrt{(x-z)(y-z^{-1})}}\left(\frac{(xy-1)^2}{(x-z)(y-z^{-1})} - xy - 3\right) \tag{8}$$

and determine the range of $\cosh 2H$ by varying z. The extremum is found to occur at $z = \sqrt{x/y}$, or $ac^3 = b^3 d$, which indeed lies in the regime $B^2 > 4AC$. This leads to the inequality $\cosh 2H > 1$. It follows that H, and hence the resulting magnetic field $L = H + 3h$, is real.

Antiferromagnetic Ising model ($K < 0$). This is the case $B^2 < 4AC$. From (3) we see that H' is real and R pure imaginary. Therefore, using (7), h is also pure imaginary. consider next the range of $\cosh 2H$. Since the extremum $z = \sqrt{xy}$ of $\cosh 2H$ determined in the above lies outside the regime $B^2 < 4AC$, a bound on $\cosh 2H$ is actually obtained by setting $B^2 = 4AC$ in (3). This consideration then leads to $|\cosh 2H| < 1$, implying H, and hence the resulting magnetic field $L = H + 3h$, is pure imaginary.

In conclusion, we have shown that the honeycomb eight-vertex model with positive vertex weights is completely equivalent to an Ising model in a non-zero magnetic field. The Ising model is either ferromagnetic with a real magnetic field, or antiferromagnetic with a magnetic field which is pure imaginary. These conclusions agree with the findings of [1, 8], but the derivation presented here is much simpler. The present approach also suggests possible extensions of our consideration. First, the method now permits straightforward extension to the asymmetric eight-vertex model, an analysis which has proven to be extremely cumbersome using the generalised weak-graph transformation [9]. Furthermore, we can also extend the analysis to other types of lattices. For a lattice of coordination number $q = 4$ such as the square lattice, the corresponding vertex model is the symmetric 16-vertex model characterised by five independent vertex weights. The analogue of (2) is therefore a set of five equations containing the four variables F, H, H', R. It then follows that the vertex model is reducible to an Ising model in a four-dimensional subspace, deduced by eliminating the four variables from the five equations. This leads to results in agreement with those previously found using the generalised weak-graph transformation [9]. Finally, we point out that all these considerations, which rely only on the fact that there exists a uniform coordination number q, hold quite generally for any lattice with the same q, regardless of the spatial dimensionality.

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